

On the uniqueness theorems of meromorphic functions with weighted sharing of three values[☆]

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Abstract

In this paper, we deal with the problem of uniqueness and weighted sharing of two meromorphic functions with their first derivatives having the same fixed points with the same multiplicities. The results in this paper improve those given by K. Tohge, Xiao-Min Li and Hong-Xun Yi.

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1. Introduction and main results

Let f and g be two non-constant meromorphic functions in the complex plane. It is assumed that the reader is familiar with the standard notations of Nevanlinna's theory such as $T(r, f)$, $m(r, f)$, $N(r, f)$, $\bar{N}(r, f)$ and so on, which can be found in [2]. We use E to denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. The notation $S(r, f)$ denotes any quantity satisfying $S(r, f) = o(T(r, f))$ ($r \rightarrow \infty$, $r \notin E$).

Let a be a complex number, we say that f and g share the value a CM provided $f - a$ and $g - a$ have the same zeros counting multiplicities (see [10]). We say that f and g share ∞ CM provided that $1/f$ and $1/g$ share 0 CM. Similarly, we say that f and g share the value a IM, provided that $f - a$ and $g - a$ have the same zeros ignoring multiplicities. In addition, we say that f and g share ∞ CM, if $1/f$ and $1/g$ share the value 0 CM, and we say that f and g share

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∞ IM, if f and g share the value 0 IM. Let $a(z)$ be a meromorphic function in the complex plane, if $T(r, a) = S(r, f)$, then $a(z)$ is called a small function of $f(z)$. In this paper, we also need the following three definitions.

Definition 1.1. (See [10, Definition 1.18].) Let f be a non-constant meromorphic function, the hyper-order of f , denoted $\nu(f)$, is defined by

$$\nu(f) = \limsup_{r \rightarrow \infty} \frac{\log \log T(r, f)}{\log r}.$$

Definition 1.2. (See [1, Definition 1].) Let p be a positive integer and $a \in C \cup \{\infty\}$. Then by $N_p(r, \frac{1}{f-a})$ we denote the counting function of those zeros of $f-a$ (counted with proper multiplicities) whose multiplicities are not greater than p , by $\bar{N}_p(r, \frac{1}{f-a})$ we denote the corresponding reduced counting function (ignoring multiplicities). By $N_{(p)}(r, \frac{1}{f-a})$ we denote the counting function of those zeros of $f-a$ (counted with proper multiplicities) whose multiplicities are not less than p , by $\bar{N}_{(p)}(r, \frac{1}{f-a})$ we denote the corresponding reduced counting function (ignoring multiplicities).

Definition 1.3. (See [3, Definition 4].) For $a \in C \cup \{\infty\}$, we put

$$\delta_p(a, f) = 1 - \limsup_{r \rightarrow \infty} \frac{N_p(r, \frac{1}{f-a})}{T(r, f)},$$

where p is a positive integer.

In 1988, K. Tohge proved the following theorem.

Theorem A. (See [9, Theorem 2].) Let f and g be two distinct transcendental meromorphic functions sharing 0, 1 and ∞ CM, and let $a (\neq 0)$ be a finite complex number. If f' and g' share a CM and $\max\{\nu(f), \nu(g)\} < 1$, then f and g satisfy one of the following relations:

- (i) $f \cdot g \equiv 1$,
- (ii) $(f-1)(g-1) \equiv 1$,
- (iii) $[(c-1)f+1] \cdot [(c-1)g-c] \equiv -c$, where $c (\neq 0, 1)$ is a constant.

Relating to Theorem A, K. Tohge [9] posed the following two questions.

Question 1.1. Is it possible to relax the hypothesis on the hyper-order of f and g ?

Question 1.2. Is it possible to weaken the restriction of CM sharing of values?

In 2002, the first question is answered by X.M. Li and H.X. Yi in the following theorem.

Theorem B. (See [4, Theorem 1].) Let f and g be two distinct transcendental meromorphic functions sharing 0, 1 and ∞ CM, and let $a (\neq 0)$ be a finite complex number. If f' and g' share a CM, then f and g satisfy one of the following relations:

- (i) $f = Ae^{a\omega z}$, $g = \frac{1}{A}e^{-a\omega z}$, where ω satisfying $\omega^2 = -1$, and $A (\neq 0)$ are constants;
- (ii) $f = 1 + Ae^{a\omega z}$, $g = 1 + \frac{1}{A}e^{-a\omega z}$, where ω satisfying $\omega^2 = -1$, and $A (\neq 0)$ are constants;

(iii) $f(z) = \frac{1}{c-1}(Ae^{a(c-1)\omega z} - 1)$, $g(z) = \frac{c}{c-1}(1 - \frac{1}{A}e^{-a(c-1)\omega z})$, where A , c and ω are constants satisfying $A \neq 0$, $c \neq 0, 1$ and $\omega^2 = \frac{1}{c}$.

In this paper, we shall deal with the above second question of K. Tohge. To this end we employ the idea of weighted sharing of values which measures how close a shared value is to being shared IM or to being shared CM. The notion is explained in the following definition.

Definition 1.4. (See [3, Definition 4].) Let k be a nonnegative integer or infinity. For any $a \in C \cup \{\infty\}$, we denote by $E_k(a, f)$ the set of all a -points of f , where an a -point of multiplicity m is counted m times if $m \leq k$, and $k+1$ times if $m > k$. If $E_k(a, f) = E_k(a, g)$, we say that f, g share the value a with weight k .

Remark 1.1. Definition 1.1 implies that if f, g share a value a with weight k , then z_0 is a zero of $f - a$ with multiplicity $m (\leq k)$ if and only if it is a zero of $g - a$ with multiplicity $m (\leq k)$, and z_0 is a zero of $f - a$ with multiplicity $m (> k)$, if and only if it is a zero of $g - a$ with multiplicity $n (> k)$, where m is not necessarily equal to n . Throughout this paper, we write f, g share (a, k) to mean that f, g share the value a with weight k . Clearly, if f, g share (a, k) , then f, g share (a, p) for all integer p , $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) , respectively.

Using the idea of weighted sharing, I. Lahiri and P. Sahoo proved the following result recently, which improved Theorem B.

Theorem C. (See [5, Theorem 1.1].) Let f and g be two distinct transcendental meromorphic functions such that f and g share $(0, 1)$, $(1, m)$ and (∞, k) , where m and k are two positive integers satisfying $(m-1)(km-1) > (1+m)^2$, and let $a (\neq 0, 1)$ be a finite complex number. If f' and g' share a CM, then f and g assume one of the relations (i)–(iii) in Theorem B.

Using the idea of weighted sharing, we can establish the following theorem, which improves Theorems A–C.

Theorem 1.1. Let f and g be two distinct transcendental meromorphic functions such that f and g share $(0, k_1)$, $(1, k_2)$ and (∞, k_3) , where k_1, k_2 and k_3 are three positive integers satisfying

$$k_1 k_2 k_3 > k_1 + k_2 + k_3 + 2. \quad (1.1)$$

If $f'(z) - z$ and $g'(z) - z$ share 0 CM, then f and g are given as one of the following three expressions:

- (i) $f(z) = Ae^{\omega z^2}$ and $g(z) = \frac{1}{A}e^{-\omega z^2}$, where $A (\neq 0)$, and ω satisfying $4\omega^2 = 1$ are two finite complex constants.
- (ii) $f(z) = 1 + Ae^{\omega z^2}$ and $g(z) = 1 + \frac{1}{A}e^{-\omega z^2}$, where $A (\neq 0)$, and ω satisfying $4\omega^2 = 1$ are two finite complex constants.
- (iii) $f(z) = \frac{Ae^{\omega z^2} - 1}{c-1}$ and $g(z) = \frac{\frac{1}{A}e^{-\omega z^2} - 1}{c-1}$, where $A (\neq 0)$, $c (\neq 0, 1)$ and ω satisfying $\omega^2 = \frac{(c-1)^2}{4c}$ are three finite complex constants.

Using proceeding as in the proof of Theorem 1.1 in Section 3 of this paper, we can prove the following theorem, which improves Theorems A–C.

Theorem 1.2. *Let f and g be two distinct transcendental meromorphic functions such that f and g share $(0, k_1)$, $(1, k_2)$ and (∞, k_3) , where k_1, k_2 and k_3 are three positive integers satisfying (1.1), and let $a (\neq 0)$ be a finite complex number. If f' and g' share a CM, then f and g assume one of the relations (i)–(iii) in Theorem B.*

From Theorems 1.1 and 1.2 we can get the following two uniqueness theorems, respectively.

Theorem 1.3. *Let f and g be two transcendental meromorphic functions such that f and g share $(0, k_1)$, $(1, k_2)$ and (∞, k_3) , where k_1, k_2 and k_3 are three positive integers satisfying (1.1). If $f'(z) - z$ and $g'(z) - z$ share 0 CM, and if the order of f is not equal to 2, then $f \equiv g$.*

Theorem 1.4. *Let f and g be two transcendental meromorphic functions such that f and g share $(0, k_1)$, $(1, k_2)$ and (∞, k_3) , where k_1, k_2 and k_3 are three positive integers satisfying (1.1), and let $a (\neq 0)$ be a finite complex number. If f' and g' share a CM, and if the order of f is not equal to 1, then $f \equiv g$.*

2. Some lemmas

Let f and g share 0, 1 and ∞ IM. We denote by $N_0(r)$ the counting function of the zeros of $f - g$ not containing the zeros of f , $\frac{1}{f}$ and $f - 1$ (see [11] or [14]).

Lemma 2.1. (See [12, Theorem 1.4].) *Let f and g be two distinct nonconstant meromorphic functions such that f and g share $(0, k_1)$, $(1, k_2)$ and (∞, k_3) , where k_1, k_2 and k_3 are three positive integers satisfying (1.1). If*

$$N_1\left(r, \frac{1}{f}\right) + N_1(r, f) < (\lambda + o(1))T(r)$$

for $r \in I$, where $I \subset [0, \infty)$ is a set such that its linear measure $\text{mes } I = \infty$, and λ is a positive integer satisfying $0 < \lambda < 1/2$, and $T(r) = \max\{T(r, f), T(r, g)\}$. Then $f \equiv g$ or $fg \equiv 1$.

Lemma 2.2. (See [13, Lemma 3].) *Let $f(z)$ be a transcendental meromorphic function, φ be a small function of f , and let n be a positive integer. Then*

$$T(r, f) < 3N\left(r, \frac{1}{f}\right) + 4N\left(r, \frac{1}{f^{(n)} - \varphi}\right) + S(r, f).$$

Lemma 2.3. (See [12, Lemma 2.6].) *Let f and g be two distinct nonconstant meromorphic functions such that f and g share $(0, k_1)$, $(1, k_2)$ and (∞, k_3) , where k_1, k_2 and k_3 are three positive integers satisfying (1.1). Then*

$$\bar{N}_2\left(r, \frac{1}{f}\right) + \bar{N}_2\left(r, \frac{1}{f-1}\right) + \bar{N}_2(r, f) = S(r, f).$$

Lemma 2.4. (See [6, Lemma 6].) *Let f and g be two distinct nonconstant meromorphic functions such that f and g share 0, 1 and ∞ IM. If f is a fractional linear transformation (Möbius transformation) of g , then f and g satisfy one of the following relations:*

- (i) $f \cdot g \equiv 1$,
- (ii) $(f - 1)(g - 1) \equiv 1$,
- (iii) $f + g \equiv 1$,
- (iv) $f \equiv cg$,
- (v) $f - 1 \equiv c(g - 1)$,
- (vi) $[(c - 1)f + 1] \cdot [(c - 1)g - c] \equiv -c$,

where $c (\neq 0, 1)$ is a finite constant.

Lemma 2.5. (See [10, Theorem 1.62].) Let f_1, f_2, \dots, f_n be nonconstant meromorphic functions, and let $f_{n+1} (\neq 0)$ be a meromorphic function such that $\sum_{i=1}^{n+1} f_i \equiv 1$. If there exists a subset $I \subseteq R^+$ satisfying $\text{mes } I = \infty$ such that

$$\sum_{i=1}^{n+1} N\left(r, \frac{1}{f_i}\right) + n \sum_{\substack{i=1 \\ i \neq j}}^{n+1} \bar{N}(r, f_i) < (\lambda + o(1))T(r, f_j) \quad (r \rightarrow \infty, r \in I, j = 1, 2, \dots, n),$$

where $\lambda < 1$, then $f_{n+1} \equiv 1$.

Lemma 2.6. (See [14, Lemma 6].) Let f_1 and f_2 be two nonconstant meromorphic functions satisfying $\bar{N}(r, f_j) + \bar{N}(r, \frac{1}{f_j}) = S(r)$ ($j = 1, 2$). Then either $\bar{N}_0(r, 1; f_1, f_2) = S(r)$ or there exist two integers s, t ($|s| + |t| > 0$) such that $f_1^s f_2^t \equiv 1$, where, and in the sequel, $\bar{N}_0(r, 1; f_1, f_2)$ denotes the reduced counting function of f_1 and f_2 related to the common 1-points and $T(r) = T(r, f_1) + T(r, f_2)$, $S(r) = o(T(r))$ ($r \rightarrow \infty, r \notin E$) only depending on f_1 and f_2 .

Lemma 2.7. (See [14, Proof of Theorems 1 and 2].) Let f and g be two distinct nonconstant meromorphic functions such that f and g share 0, 1 and ∞ CM, and let $N_0(r) \neq S(r, f)$. If f is a fractional linear transformation of g , then $N_0(r) = T(r, f) + S(r, f)$. If f is not any fractional linear transformation of g , then $N_0(r) \leq \frac{1}{2}T(r, f) + S(r, f)$, and f and g assume one of the following relations:

- (i) $f \equiv \frac{e^{(k+1)\gamma} - 1}{e^{s\gamma} - 1}, g \equiv \frac{e^{-(k+1)\gamma} - 1}{e^{-s\gamma} - 1}$;
- (ii) $f \equiv \frac{e^{s\gamma} - 1}{e^{(k+1)\gamma} - 1}, g \equiv \frac{e^{-s\gamma} - 1}{e^{-(k+1)\gamma} - 1}$;
- (iii) $f \equiv \frac{e^{s\gamma} - 1}{e^{-(k+1-s)\gamma} - 1}, g \equiv \frac{e^{-s\gamma} - 1}{e^{(k+1-s)\gamma} - 1}$;

where γ is a nonconstant entire function, s and k (≥ 2) are positive integers such that s and $k + 1$ are mutually prime and $1 \leq s \leq k$.

Lemma 2.8. (See [10, Proof of Theorems 1.12 and 1.13].) Let f be a nonconstant meromorphic function, and let $F = \sum_{k=0}^p a_k f^k / \sum_{j=0}^q b_j f^j$ be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_p \neq 0$ and $b_q \neq 0$. Then $T(r, F) = dT(r, f) + O(1)$, where $d = \max\{p, q\}$.

Lemma 2.9. (See [7, Lemma 2.5].) Let s (> 0) and t are mutually prime integers, and let c be a finite complex number such that $c^s = 1$, then there exists one and only one common zero of $\omega^s - 1$ and $\omega^t - c$.

Lemma 2.10. (See [5, Proof of Lemma 2.1].) Let f and g be two distinct nonconstant meromorphic functions such that f and g share $(0, 1)$, $(1, m)$ and (∞, k) , where $k (\geq 0)$ and $m (\geq 1)$ are two integers, and let $a \in \{0, 1\}$. If $h'_1 h' \equiv -z^2(h_1 h - 1)^2$, where $h_1 = (f - 1)/(g - 1)$ and $h = g/f$, then f and g share ∞ CM, and every common a -point $z_a (\neq 0)$ of f and g has the same multiplicity.

Lemma 2.11. (See [8, Lemma 2.4].) Let h be a nonconstant meromorphic function, and let α, β, γ be meromorphic functions such that $T(r, \alpha) + T(r, \beta) + T(r, \gamma) = S(r, h)$, where $\alpha \not\equiv 0$ or $\gamma \not\equiv 0$. Furthermore, let $H = \alpha h^2 + \beta h + \gamma$. If

$$\bar{N}(r, h) + \bar{N}\left(r, \frac{1}{h}\right) + N_1\left(r, \frac{1}{H}\right) = S(r, h),$$

then $\beta^2 - 4\alpha\gamma \equiv 0$.

3. Proof of theorem

Proof of Theorem 1.1. Let

$$\frac{f' - z}{g' - z} = H. \quad (3.1)$$

Suppose that $H \equiv A_1$, where A_1 is a nonzero constant. From (3.1) we get

$$2(f - A_1 g) = (1 - A_1)z^2 + B_1, \quad (3.2)$$

where B_1 is a constant. Since $f \not\equiv g$, from (3.2) we know that

$$(1 - A_1)z^2 + B_1 \not\equiv 0. \quad (3.3)$$

By (3.2) and (3.3) we get

$$\delta_1(0, f) + \delta_1(1, f) = 2. \quad (3.4)$$

Let

$$F = \frac{f - 1}{f} \quad \text{and} \quad G = \frac{g - 1}{g}. \quad (3.5)$$

From (3.5) and the assumptions of Theorem 1.1 we can see that F and G share $(0, k_2)$, $(1, k_3)$ and (∞, k_1) . So from (3.4) and (3.5) we deduce

$$\delta_1(0, F) + \delta_1(\infty, F) = 2,$$

which implies that

$$N_1\left(r, \frac{1}{F}\right) + N_1(r, F) < (\lambda + o(1))T(r, F) \quad (3.6)$$

for $r \in I$, where $I \subset [0, \infty)$ is a set such that its linear measure $\text{mes } I = \infty$, and λ is a positive integer satisfying $0 < \lambda < 1/2$. From (3.6) and Lemma 2.1 we get $F \equiv G$ or $FG \equiv 1$. If $F \equiv G$, then it follows from (3.5) we deduce $f \equiv g$, this contradicts the assumption of Theorem 1.1. If $FG \equiv 1$, then from (3.5) we deduce

$$f + g \equiv 1 \quad (3.7)$$

and

$$f' + g' \equiv 0. \quad (3.8)$$

From (3.7) we deduce that 0 is a Picard value of f . Noting that f is a transcendental meromorphic function, from (3.8), Lemma 2.2 and the condition that $f'(z) - z$ and $g'(z) - z$ share 0 CM, we can get a contradiction. Thus H is not a constant, and hence

$$\frac{H'}{H} \not\equiv 0. \quad (3.9)$$

By logarithmic differentiation, from (3.1) we obtain

$$\frac{H'}{H} = \frac{f'' - 1}{f' - z} - \frac{g'' - 1}{g' - z}. \quad (3.10)$$

By (3.1), (3.9), (3.10), the assumptions of Theorem 1.1 and Lemma 2.3 we get

$$\begin{aligned} N_1(r, f) &\leq N\left(r, \frac{H}{H'}\right) \leq T\left(r, \frac{H'}{H}\right) + O(1) \\ &= m\left(r, \frac{H'}{H}\right) + \bar{N}(r, H) + \bar{N}\left(r, \frac{1}{H}\right) + O(1) \\ &\leq m\left(r, \frac{H'}{H}\right) + \bar{N}_{(2)}(r, f) + \bar{N}_{(2)}(r, g) + O(1) = S(r, f), \end{aligned}$$

namely

$$N_1(r, f) = S(r, f). \quad (3.11)$$

By Lemma 2.3 we have

$$\bar{N}_{(2)}(r, f) = S(r, f). \quad (3.12)$$

By (3.11) and (3.12) we obtain

$$\bar{N}(r, f) = S(r, f). \quad (3.13)$$

We discuss the following two cases.

Case 1. Suppose that f is a Möbius transformation of g . If f and g satisfy the relation (iii) in Lemma 2.4, from the condition that f and g share 0 and 1 IM we can see that 0 and 1 are Picard exceptional values of f , and so it follows by the second fundamental theorem that

$$T(r, f) = \bar{N}(r, f) + S(r, f), \quad (3.14)$$

which contradicts (3.13). If f and g satisfy the relation (iv) in Lemma 2.4, from the condition that f and g share 1 IM we can see that 1 and c are Picard exceptional values of f . Combining the second fundamental theorem we immediately yields (3.14), this contradicts (3.13). If f and g satisfy the relation (v) in Lemma 2.4, then 0 is a Picard exceptional value and $f' \equiv cg'$. Combining the condition that $f'(z) - z$ and $g'(z) - z$ share 0 CM, we can deduce

$$N\left(r, \frac{1}{f'(z) - z}\right) = O(\log r). \quad (3.15)$$

Noting that f is a transcendental meromorphic function, from (3.15) and Lemma 2.2 we can deduce $T(r, f) = S(r, f)$, this is impossible. Thus f and g satisfy one of the relations (i), (ii) and (vi) in Lemma 2.4. Assume that f and g satisfy the relation (i) in Lemma 2.4, then

$$f = e^\gamma \quad \text{and} \quad g = e^{-\gamma}, \quad (3.16)$$

where γ is a nonconstant entire function. From (3.16) we deduce

$$f' = \gamma' e^\gamma \quad \text{and} \quad g' = -\gamma' e^{-\gamma}. \quad (3.17)$$

Substituting (3.17) into (3.1) and noting that f and g share ∞ CM, we get

$$\frac{\gamma' e^{2\gamma} - z e^\gamma}{-\gamma' - z e^\gamma} = H =: e^\delta, \quad (3.18)$$

where, and in the sequel, δ is a nonconstant entire function. From (3.18) we deduce

$$T(r, e^\delta) \geq T(r, e^\gamma) + S(r, f). \quad (3.19)$$

Since (3.18) can be rewritten as

$$\gamma' e^\gamma - z e^\delta + \gamma' e^{\delta-\gamma} \equiv z. \quad (3.20)$$

By (3.20) and Lemma 2.5 we deduce

$$\gamma' e^\gamma - z e^\delta \equiv 0, \quad \gamma' e^{\delta-\gamma} \equiv z. \quad (3.21)$$

From (3.21) we deduce

$$(\gamma')^2 = z^2,$$

and so

$$\gamma = \omega z^2 + b, \quad (3.22)$$

where ω satisfying $4\omega^2 = 1$, and b are finite complex constants. Substituting (3.22) into (3.16) we get the conclusion (i) of Theorem 1.1. Assume that f and g satisfy the relation (ii) in Lemma 2.4, then

$$f = 1 + e^\gamma \quad \text{and} \quad g = 1 + e^{-\gamma}, \quad (3.23)$$

where γ is a nonconstant entire function. From (3.23) and in the same manner as above we easily deduce the conclusion (ii) of Theorem 1.1. Assume that f and g satisfy the relation (vi) in Lemma 2.4, then

$$f = \frac{e^\gamma - 1}{c - 1} \quad \text{and} \quad g = \frac{e^{-\gamma} - 1}{c^{-1} - 1}, \quad (3.24)$$

where γ is a nonconstant entire function, and $c (\neq 0, 1)$ is some finite complex constant. From (3.24) we deduce

$$f' = \frac{\gamma' e^\gamma}{c - 1} \quad \text{and} \quad g' = \frac{-\gamma' e^{-\gamma}}{c^{-1} - 1}. \quad (3.25)$$

Substituting (3.25) into (3.1) we get

$$\frac{\frac{\gamma'}{c-1} \cdot e^{2\gamma} - z e^\gamma}{\frac{-\gamma'}{c^{-1}-1} - z e^\gamma} = e^\delta. \quad (3.26)$$

By (3.26) we have

$$T(r, e^\delta) \geq T(r, e^\gamma) + S(r, f). \quad (3.27)$$

Since (3.26) can be rewritten as

$$\frac{\gamma'}{(c-1)z} \cdot e^\gamma + e^\delta + \frac{\gamma'}{(c^{-1}-1)z} \cdot e^{\delta-\gamma} \equiv 1, \quad (3.28)$$

from (3.27), (3.28) and Lemma 2.5 we obtain

$$\frac{\gamma'}{(c^{-1}-1)z} \cdot e^{\delta-\gamma} \equiv 1, \quad \frac{\gamma'}{(c-1)z} \cdot e^\gamma + e^\delta \equiv 0. \quad (3.29)$$

From (3.29) we get

$$\gamma = \omega z^2 + b, \quad (3.30)$$

where ω satisfying $\omega^2 = \frac{(c-1)^2}{4c}$ and b are finite complex constants. Substituting (3.30) into (3.24) we get the conclusion (iii) of Theorem 1.1.

Case 2. Suppose that f is not any Möbius transformation of g . Let

$$\frac{f-1}{g-1} = h_1, \quad \frac{f}{g} = h_2 \quad (3.31)$$

and

$$h_0 = \frac{h_1}{h_2}, \quad (3.32)$$

where h_0, h_1 and h_2 are three nonconstant meromorphic functions satisfying

$$T(r, g) + T(r, h_1) + T(r, h_2) = O(T(r, f)) \quad (r \notin E). \quad (3.33)$$

Noting that f and g share $(0, k_1)$, $(1, k_2)$ and (∞, k_3) , from Lemma 2.3 we deduce

$$\overline{N}(r, h_j) + \overline{N}\left(r, \frac{1}{h_j}\right) = S(r, f) \quad (j = 0, 1, 2). \quad (3.34)$$

From (3.31) and (3.32) we have

$$f = \frac{h_1 - 1}{h_0 - 1} \quad \text{and} \quad g = \frac{h_1^{-1} - 1}{h_0^{-1} - 1}, \quad (3.35)$$

and

$$\frac{(f-1)g}{f(g-1)} \equiv h_0. \quad (3.36)$$

From (3.35) we obtain

$$f - g = \frac{(h_1 - 1)(1 - h_0 h_1^{-1})}{h_0 - 1}. \quad (3.37)$$

Then from (3.31), (3.32), (3.34), (3.35) and (3.37) we easily deduce

$$N_0(r) = \overline{N}_0(r, 1; h_1, h_0) + S(r, f) = \overline{N}_0(r, 1; h_1, h_2) + S(r, f). \quad (3.38)$$

We discuss the following two subcases.

Subcase 2.1. Suppose that

$$N_0(r) \neq S(r, f). \quad (3.39)$$

Then from (3.38) and (3.39) we get

$$\bar{N}_0(r, 1; h_1, h_2) \neq S(r, f). \quad (3.40)$$

By (3.40) and Lemma 2.6 we know that there exist two integers s and t ($|s| + |t| > 0$) such that

$$h_1^s h_2^t \equiv 1. \quad (3.41)$$

Substituting (3.31) into (3.41) we get

$$f^t(f-1)^s \equiv g^t(g-1)^s. \quad (3.42)$$

Noting that f is not any Möbius transformation of g , from (3.42) we deduce that $s \neq 0$, $t \neq 0$ and $|s| \neq |t|$, and so we deduce that f and g share 0, 1 and ∞ CM. Thus from (3.39) and Lemma 2.7 we have

$$0 < \limsup_{\substack{r \rightarrow \infty \\ r \notin E}} \frac{N_0(r)}{T(r, f)} \leq \frac{1}{2}, \quad (3.43)$$

and that f and g assume one of the three relations (i)–(iii) in Lemma 2.7. Assume that f and g satisfy the relation (ii) in Lemma 2.7, from Lemma 2.8 and Lemma 2.9 we deduce $\bar{N}(r, f) = T(r, f) + S(r, f)$, which contradicts (3.13). Assume that f and g satisfy the relation (i) in Lemma 2.7, by (3.13) we know that $s = 1$, and

$$f = e^{k\gamma} + e^{(k-1)\gamma} + \cdots + 1, \quad g = e^{-k\gamma} + e^{-(k-1)\gamma} + \cdots + 1. \quad (3.44)$$

By (3.44) we obtain

$$T(r, f) = kT(r, e^\gamma) + S(r, f), \quad T(r, g) = kT(r, e^\gamma) + S(r, f). \quad (3.45)$$

Substituting (3.44) into (3.1) and noting that $H = e^\delta$, we get

$$\frac{k\gamma' e^{2k\gamma} + (k-1)\gamma' e^{(2k-1)\gamma} + \cdots + \gamma' e^{(k+1)\gamma} - ze^{k\gamma}}{-k\gamma' - (k-1)\gamma' e^\gamma - \cdots - \gamma' e^{(k-1)\gamma} - ze^{k\gamma}} = e^\delta. \quad (3.46)$$

By (3.45) and (3.46) we have

$$T(r, e^\delta) \geq kT(r, e^\gamma) + S(r, f) \quad (3.47)$$

and

$$\frac{k\gamma'}{z} e^{k\gamma} + \frac{(k-1)\gamma'}{z} e^{(k-1)\gamma} + \cdots + \frac{\gamma'}{z} e^\gamma + e^\delta + \frac{\gamma'}{z} e^{\delta-\gamma} + \frac{2\gamma'}{z} e^{\delta-2\gamma} + \cdots + \frac{k\gamma'}{z} e^{\delta-k\gamma} \equiv 1. \quad (3.48)$$

By Lemma 2.5, (3.45), (3.47) and (3.48) we obtain $\frac{k\gamma'}{z} e^{\delta-k\gamma} \equiv 1$, which reads

$$e^\delta \equiv \frac{z}{k\gamma'} e^{k\gamma}. \quad (3.49)$$

Substituting (3.49) into (3.48) we get

$$\left(\frac{k\gamma'}{z} + \frac{z}{k\gamma'}\right) e^{k\gamma} + \left(\frac{(k-1)\gamma'}{z} + \frac{1}{k}\right) e^{(k-1)\gamma} + \cdots + \left(\frac{\gamma'}{z} + \frac{k-1}{k}\right) e^\gamma \equiv 0. \quad (3.50)$$

From (3.50) we obtain

$$\frac{k\gamma'}{z} + \frac{z}{k\gamma'} \equiv 0, \quad \frac{(k-1)\gamma'}{z} + \frac{1}{k} \equiv 0, \quad \frac{\gamma'}{z} + \frac{k-1}{k} \equiv 0. \quad (3.51)$$

From (3.51) we have a contradiction. Assume that f and g satisfy the relation (iii) in Lemma 2.7, by (3.13) we know that $s = k$. Thus

$$f = -e^{k\gamma} - e^{(k-1)\gamma} - \dots - e^\gamma, \quad g = -e^{-k\gamma} - e^{-(k-1)\gamma} - \dots - e^{-\gamma}.$$

In the same manner as above, we can get a contradiction.

Subcase 2.2. Assume

$$N_0(r) = S(r, f). \quad (3.52)$$

By (3.35), (3.38) and (3.52), we have

$$\bar{N}(r, f) = \bar{N}\left(r, \frac{1}{h_0 - 1}\right) + S(r, f). \quad (3.53)$$

By (3.13), (3.34), (3.53) and the second fundamental theorem we deduce

$$T(r, h_0) = S(r, f). \quad (3.54)$$

From (3.33), (3.35) and (3.54) we have

$$T(r, f) = T(r, h_1) + S(r, f), \quad T(r, g) = T(r, f) + S(r, f). \quad (3.55)$$

Let $h = 1/h_2$. Combining (3.32) and (3.35) we can get

$$h_0 = h_1 h \quad (3.56)$$

and

$$f = \frac{h_1 - 1}{h_1 h - 1}, \quad g = \frac{h(h_1 - 1)}{h_1 h - 1}. \quad (3.57)$$

Substituting (3.57) into (3.1) we can get

$$H = \frac{A_2 h_1 + B_2}{C_2 h + D_2} = \frac{A_2 h_1^2 + B_2 h_1}{C_3 + D_2 h_1}, \quad (3.58)$$

where

$$A_2 = \frac{h_1'}{h_1} \cdot (h_1 h - 1) - (h_1 h)', \quad B_2 = (h_1 h)' - z(h_1 h - 1)^2, \quad (3.59)$$

$$C_2 = (h_1 h)' - \frac{h'}{h} \cdot (h_1 h - 1), \quad C_3 = C_2 h_1 h \quad (3.60)$$

and

$$D_2 = -(h_1 h)' - z(h_1 h - 1)^2. \quad (3.61)$$

Noting that f is a transcendental meromorphic function, from (3.34), (3.54)–(3.56) and (3.59)–(3.61) we can deduce

$$T(r, A_2) + T(r, B_2) + T(r, C_2) + T(r, C_3) + T(r, D_2) = S(r, f). \quad (3.62)$$

If $C_3 \equiv 0$, noting that h_0 is a nonconstant meromorphic function, from (3.56) and (3.60) we can get $C_2 \equiv 0$, and so $h'/h = h_0'/(h_0 - 1)$. Combining (3.56) we can deduce $h_1 = (C_4 h_0)/(h_0 - 1)$,

where $C_4 (\neq 0)$ is a finite complex constant. From this and (3.54) we deduce $T(r, h_1) = S(r, f)$, which contradicts (3.55). Thus

$$C_3 \neq 0. \quad (3.63)$$

Similarly

$$A_2 \neq 0. \quad (3.64)$$

By (3.55), (3.58), (3.62)–(3.64) we can deduce

$$T(r, H) \geq T(r, h_1) + S(r, f). \quad (3.65)$$

If $B_2 \equiv 0$ and $D_2 \equiv 0$, it follows by (3.56), (3.59) and (3.61) that $h'_0 \equiv 0$, and so there is a finite complex constant A_3 such that $h_0 \equiv A_3$, this is impossible. Thus B_2 and D_2 can not be together identically zero. If $B_2 \equiv 0$ and $D_2 \neq 0$, from (3.58) we can get

$$A_2 h_1^2 = (C_3 + D_2 h_1) H. \quad (3.66)$$

From (3.62)–(3.64), (3.66) and by noting $\bar{N}(r, H) + N(r, 1/H) = S(r, f)$ we deduce $-\frac{C_3}{D_2} \neq 0, \infty$, and

$$T\left(r, -\frac{C_3}{D_2}\right) = S(r, f), \quad \bar{N}\left(r, \frac{1}{h_1 + \frac{C_3}{D_2}}\right) = S(r, f). \quad (3.67)$$

From (3.34), (3.55), (3.67) and Nevanlinna's three small functions theorem (see [10, Theorem 1.36]) we can get

$$T(r, h_1) < \bar{N}(r, h_1) + \bar{N}\left(r, \frac{1}{h_1}\right) + \bar{N}\left(r, \frac{1}{h_1 + \frac{C_3}{D_2}}\right) = S(r, f),$$

which contradicts (3.55). If $D_2 \equiv 0$ and $B_2 \neq 0$, from (3.58) we get

$$C_3 H = (A_2 h_1 + B_2) \cdot h_1. \quad (3.68)$$

From (3.62)–(3.64) and (3.68), in the same manner as above we can get a contradiction. If $B_2 \neq 0$ and $D_2 \neq 0$, from (3.56) and (3.58) we can get

$$-\frac{A_2}{B_2} \cdot h_1 + \frac{D_2}{B_2} \cdot H + \frac{C_2}{B_2} \cdot h H \equiv 1. \quad (3.69)$$

Noting that $h = 1/h_2$ and $\bar{N}(r, H) + \bar{N}(r, 1/H) = S(r, f)$, from (3.34), (3.55), (3.62), (3.65), (3.69) and Lemma 2.5 we deduce

$$\frac{C_2}{B_2} \cdot h H \equiv 1 \quad (3.70)$$

and $-\frac{A_2}{B_2} \cdot h_1 + \frac{D_2}{B_2} \cdot H \equiv 0$. Thus

$$h_1 h A_2 C_2 - B_2 D_2 \equiv 0. \quad (3.71)$$

Substituting (3.59)–(3.61) into (3.71) we have

$$\begin{aligned} (h_1 h) \cdot \left(\frac{h'_1}{h_1} \cdot (h_1 h - 1) - (h_1 h)' \right) \cdot \left((h_1 h)' - \frac{h'}{h} \cdot (h_1 h - 1) \right) \\ + ((h_1 h)' - z(h_1 h - 1)^2) \cdot ((h_1 h)' + z(h_1 h - 1)^2) \equiv 0, \end{aligned}$$

from which we can deduce

$$(h'_1(h_1h - 1) - h_1(h_1h)')(h(h_1h)' - h'(h_1h - 1)) + ((h_1h)')^2 - z^2(h_1h - 1)^4 \equiv 0,$$

i.e.

$$-((h_1h)')^2(h_1h - 1) + (h'_1h + h_1h')(h_1h)'(h_1h - 1) - h'_1h'(h_1h - 1)^2 - z^2(h_1h - 1)^4 \equiv 0,$$

which implies that

$$h'_1h'(h_1h - 1)^2 + z^2(h_1h - 1)^4 \equiv 0. \quad (3.72)$$

Noting that h_0 is a nonconstant meromorphic function, from (3.56) we can see that $h_1h \not\equiv 1$, and so it follows from (3.72) that

$$h'_1h' \equiv -z^2(h_1h - 1)^2. \quad (3.73)$$

From (3.56) we have

$$h = \frac{h_0}{h_1}. \quad (3.74)$$

Substituting (3.56) and (3.74) into (3.73) we deduce

$$h_0 \left(\alpha_1 - \frac{\alpha_0}{2} \right)^2 \equiv z^2 h_0^2 + \left(\frac{\alpha_0^2}{4} - 2z^2 \right) h_0 + z^2, \quad (3.75)$$

where $\alpha_j = h'_j/h_j$ ($j = 0, 1$). We discuss the following two subcases.

Subcase 2.2.1. Suppose that h_0 is a transcendental meromorphic function. From (3.1), (3.36), (3.73) and Lemma 2.10 we can deduce $H = e^\delta$ and $h_0 = A_3 z^{s_1} e^{\delta_1}$, where $A_3 (\neq 0)$ is a finite complex constant, s_1 is an integer, and δ_1 is a nonconstant entire function. Thus

$$N_{(1)} \left(r, \frac{1}{h_0} \right) + T(r, z^2) + T \left(r, \frac{\alpha_0^2}{4} - 2z^2 \right) = S(r, h_0). \quad (3.76)$$

From (3.75), (3.76) and Lemma 2.11 we can get

$$\left(\frac{\alpha_0^2}{4} - 2z^2 \right)^2 - 4z^4 \equiv 0. \quad (3.77)$$

From (3.77) we get

$$\alpha_0 = \frac{h'_0}{h_0} = 4z\omega, \quad h_0 = Ae^{2\omega z^2}, \quad (3.78)$$

where ω satisfying $\omega^2 = 1$, and A are nonzero constants. Substituting (3.78) into (3.75) we deduce

$$\alpha_1 = \frac{h'_1}{h_1} = 2z\omega + z\omega_1 \left(B_0 e^{\omega z^2} + \frac{1}{B_0} e^{-\omega z^2} \right),$$

where B_0 and ω_1 are constants satisfying $B_0^2 = A$ and $\omega_1^2 = 1$. Thus

$$\ln h_1 = \omega z^2 + \frac{\omega_1}{2\omega} \left(B_0 e^{\omega z^2} - \frac{1}{B_0} e^{-\omega z^2} \right) + C, \quad (3.79)$$

where C is a constant. Set $B = \frac{\omega_1 B_0}{\omega}$, then $B^2 = A$ and

$$\ln h_1 = \omega z^2 + \frac{1}{2} \left(B e^{\omega z^2} - \frac{1}{B} e^{-\omega z^2} \right) + C. \quad (3.80)$$

From (3.80) we deduce

$$\alpha_1 = \frac{h'_1}{h_1} = 2z\omega + z\omega \left(B e^{\omega z^2} + \frac{1}{B} e^{-\omega z^2} \right). \quad (3.81)$$

Noting that $B^2 = A$, from (3.59), (3.60), (3.78) and (3.81) we deduce

$$B_2 = \alpha_0 h_0 - z h_0^2 + 2z h_0 - z = (4z\omega + 2z) B^2 e^{2z^2\omega} - z B^4 e^{4z^2\omega} - z \quad (3.82)$$

and

$$\begin{aligned} C_3 &= \alpha_1 h_0^2 + \alpha_0 h_0 - \alpha_1 h_0 \\ &= B^5 z \omega e^{5z^2\omega} + 2B^4 z \omega e^{4z^2\omega} + 2B^2 z \omega e^{2z^2\omega} - B z \omega e^{z^2\omega}. \end{aligned} \quad (3.83)$$

On the other hand, from (3.56), (3.59), (3.60), (3.74) and the condition $H = e^\delta$ we can deduce that (3.70) can be rewritten as

$$\frac{e^\delta}{h_1} = \frac{B_2}{C_3} = \frac{h'_0 - z h_0^2 + 2z h_0 - z}{\alpha_1 h_0^2 + h'_0 - \alpha_1 h_0}. \quad (3.84)$$

Substituting (3.82) and (3.83) into (3.84) we deduce

$$\frac{e^{4z^2\omega} - \frac{(4\omega+2)}{B^2} \cdot e^{2z^2\omega} + \frac{1}{B^4}}{e^{4z^2\omega} + \frac{2}{B} e^{3z^2\omega} + \frac{2}{B^3} e^{z^2\omega} - \frac{1}{B^4}} \equiv - \frac{B \omega e^{\delta+z^2\omega}}{h_1}. \quad (3.85)$$

Let

$$P_1(\chi) = \chi^4 - \frac{4\omega+2}{B^2} \chi^2 + \frac{1}{B^4}, \quad P_2(\chi) = \chi^4 + \frac{2}{B} \chi^3 + \frac{2}{B^3} \chi - \frac{1}{B^4}. \quad (3.86)$$

From (3.86) we can see that every root of $P_j(\chi) = 0$ ($j = 1, 2$) is not equal to zero, and that there is at least one root of $P_1(\chi) = 0$ that is not any root of $P_2(\chi) = 0$. From (3.31), (3.36) and (3.78) we can deduce that 0 and ∞ are two Picard exceptional values of h_1 , and so from (3.85) we can have a contradiction.

Subcase 2.2.2. Suppose that h_0 is a nonconstant rational function. From (3.1), (3.36), (3.73) and Lemma 2.10 we can deduce $H = e^\delta$ and

$$h_0 = A_4 \cdot z^{s_2}, \quad (3.87)$$

where $A_4 (\neq 0)$ is a finite complex constant, and s_2 is a nonzero integer. If $s_2 > 0$ and $z = 0$ is a zero of h_1 , from Lemma 2.10 we can see that f and g share 0 and ∞ CM. Combining the condition that h_1 is a transcendental meromorphic function, from (3.31), (3.36), (3.56) and (3.87) we can deduce $h = g/f = e^{-\delta_3}$ and

$$h_1 = A_4 \cdot z^{s_2} e^{\delta_3}, \quad (3.88)$$

where δ_3 is a nonconstant entire function. Substituting (3.87) and (3.88) into (3.75) and (3.84) we can deduce

$$\alpha_1^2 = A_4 \cdot z^{s_2+2}(1 + o(1)),$$

and so

$$e^{2(\delta-\delta_3)} = A_4 \cdot z^{s_2}(1 + o(1)), \quad (3.89)$$

as $z \rightarrow \infty$. From (3.89) we can get $s_2 = 0$. This is impossible. If $s_2 > 0$ and $z = 0$ is a zero of $h = g/f$, from Lemma 2.10 we can see that f and g share 1 and ∞ CM, and so it follows from (3.31), (3.36), (3.56) and (3.87) that $h_1 = e^{\delta_4}$ and $h = A_4 \cdot z^{s_2} e^{-\delta_4}$, where δ_4 is a nonconstant entire function. Next in the same manner as above, from (3.75) and (3.84) we can get $s_2 = 0$, this is impossible. If $s_2 < 0$, using proceeding as above we can get contradictions.

Theorem 1.1 is thus completely proved. \square

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